To obtain the desired vacuum chamber pressure history, predetermined pressure profiles were used on a Moseley X-Y recorder. Three evacuation valves were manually operated so that the recorder pen followed the plotted curve during pumpdown (Fig. 2). At the higher pumpdown rates, it usually took several tries before an acceptable chamber pressure history was obtained. Times to evacuate the vacuum chamber from ambient pressure to 0.25 psia ranged from 105 to 3.5 sec, and the transducer output signals were recorded on a high-frequency Honeywell oscillograph.

Test Results and Correlation with Theory

The data from a typical run are shown in Fig. 4. This graph shows the actual and nominal vacuum chamber pressure histories together with the output of the two differential pressure transducers. Also shown are the theoretical predictions from the solution to Eqs. (4) and (5). A value of the permeability K was chosen so that the solution would match the value of the maximum measured pressure differential at the no-flow boundary. In Fig. 4, the theoretical pressure differential curve for the location 19 in. downstream of the no-flow boundary was generated by using the same value of K chosen for the NFB case. The viscosity was always evaluated at the average of the chamber and tank temperatures. The value of K for a specific layer density was then obtained by averaging the values for each run. This was done extensively for the 30 layers/in. density where four different evacuation lengths, four pumpdown rates, and three mean temperatures were used. The following values were derived for the two-layer densities: $K = 1.47 \times 10^{-6}\text{ft}^2$, for Superfloc at 30 layers/in. and $K = 5.01 \times 10^{-7}\text{ft}^2$, for Superfloc at 45 layers/in.

For design purposes, the maximum pressure differential is usually the parameter of interest. Figure 5 presents these data as a function of evacuation length and ambient pressure history. Figure 6 shows the effect of viscosity (temperature). It can be observed that the model does not correlate well over a wide temperature range (280°–535°R). It should be noted that all test points were used in the calculation of the permeability K. If the points at the higher mean temperature (535°R) were deleted, the correlations of $\Delta P_{\max}$ vs L presented in Fig. 5 would be closer.

Derivation of the Quaternion Scheme via the Euler Axis and Angle

Carl Grubin*
Hughes Aircraft Company, Culver City, Calif.

A QUATERNION is a four-parameter system for uniquely specifying the attitude of a rigid body with respect to some reference frame. The four parameters themselves are determined by solving the quaternion-rate equations, which are linear differential equations whose coefficients are the body angular rates. Once the parameters are determined, the elements of the direction-cosine matrix that determine the body attitude can be formulated. In this Note the direction cosine matrix in terms of the quaternion parameters and the quaternion-rate equations are both derived in a simple physical way using the concept of the Euler axis and angle. A supplementary result of these derivations is an explicit solution for the initial quaternion in terms of the initial direction cosines.

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* Senior Staff Engineer, Aeronautical Systems Division. Associate Fellow, AIAA.

Analysis

A theorem due to Euler states (in effect) that any sequence of rotations of a rigid body which has one point fixed can be arrived at by a single rotation—the Euler angle (Author's nomenclature)—about some axis—the Euler axis (Author's nomenclature)—which passes through the fixed point.[1] Let the fixed point in the body be located at the origin of a set of fixed Cartesian axes X, Y, Z, with unit vectors $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$. Introduce a second set of Cartesian axes x, y, z, with unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, which are fixed in the body. Let the angular velocity of the body at any time, $\boldsymbol{\omega}$, be described by components p, q, r measured along x, y, z. Then

$$\boldsymbol{\omega} = \mathbf{i}p + \mathbf{j}q + \mathbf{k}r \tag{1}$$

where p, q, r are assumed to be known time functions.

Now the time history of p, q, r integrated from $t = 0$ to the present time $t = t$ has brought the rigid body from the reference attitude—x, y, z coincident with X, Y, Z—to its present attitude. By Euler's theorem this sequence of motions is equivalent to a single rotation θ about some well-defined axis which passes through the origin. Let this axis be defined by a unit vector $\mathbf{e}_\theta$, with the rotation θ around $\mathbf{e}_\theta$ taken positive in the sense of the right hand convention. Initially, the viewpoint is that $\mathbf{e}_\theta$ and θ are known and the first problem is to calculate the transformation matrix between X, Y, Z and x, y, z. Later, equations are developed for determining $\mathbf{e}_\theta$ and θ given the time history of p, q, r.

The transformation matrix can be obtained by applying a general formula for rigid-body rotation derived earlier.[2] Using this formula one can determine the final orientation of any line in a rigid body, which intersects a fixed axis, when the rigid body undergoes a finite rotation about this axis (Fig. 1). The result is

$$\mathbf{r}_f = \mathbf{r}_0(\cos\theta) + \mathbf{e}_\theta(\mathbf{e}_\theta\cdot\mathbf{r}_0)(1 - \cos\theta) + (\mathbf{e}_\theta \times \mathbf{r}_0)\sin\theta \tag{2}$$

where $\mathbf{e}_\theta$ = unit vector along the axis of rotation, θ = rotation angle, and $\mathbf{r}_0$, $\mathbf{r}_f$ = initial and final orientations of the line.

In the present paper the axis-angle combination is again designated by $\mathbf{e}_\theta$ and θ, so Eq. (2) can be applied directly by substituting the appropriate axes for $\mathbf{r}_0$ and $\mathbf{r}_f$. Thus, to find the direction cosines for unit vector $\mathbf{i}$, take

$$\mathbf{r}_f = \mathbf{i} \text{ and } \mathbf{r}_0 = \mathbf{i}_0 = \mathbf{I} \tag{3}$$

since at the start of every rotation x is coincident with X.

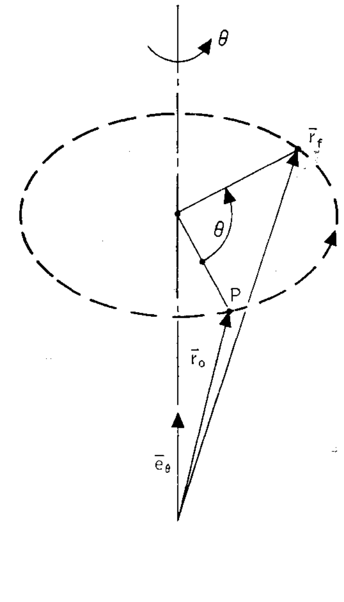

Fig. 1 Finite rotation.

Further, let the rotation axis $\mathbf{e}_\theta$ be defined by direction cosines ℓ, m, n measured with respect to X, Y, Z. So

$$\mathbf{e}_\theta = \mathbf{I}\ell + \mathbf{J}m + \mathbf{K}n \tag{4}$$

Substituting Eqs. (3) and (4) into Eq. (2)

$$\mathbf{i} = \mathbf{I}\cos\theta + (\mathbf{I}\ell + \mathbf{J}m + \mathbf{K}n)\ell(1 - \cos\theta) + (\mathbf{I}\ell + \mathbf{J}m + \mathbf{K}n) \times \mathbf{I}\sin\theta \tag{5}$$

Equation (5) gives the direction cosines of body axis x with respect to the fixed axes X, Y, Z. Proceeding similarly for $\mathbf{j}$ and $\mathbf{k}$, take

$$\mathbf{r}_f = \mathbf{j}, \mathbf{r}_0 = \mathbf{j}_0 = \mathbf{J} \tag{6a}$$

$$\mathbf{r}_f = \mathbf{k}, \mathbf{r}_0 = \mathbf{k}_0 = \mathbf{K} \tag{6b}$$

with $\mathbf{e}_\theta$ always given by Eq. (4). The complete transformation matrix is then given by Eq. (7).

$$\begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} C + \ell^2(1 - C) & \ell m(1 - C) + nS & \ell n(1 - C) - mS \\ \ell m(1 - C) - nS & C + m^2(1 - C) & mn(1 - C) + \ell S \\ \ell n(1 - C) + mS & mn(1 - C) - \ell S & C + n^2(1 - C) \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{bmatrix}$$

where

$$C \equiv \cos\theta, S \equiv \sin\theta \tag{7}$$

Equation (7) is the first principal result; it is the transformation between the body-fixed system and the reference system expressed in terms of the direction cosines of the Euler axis ℓ, m, n, and the Euler angle θ. When these four parameters are known, the transformation is known. The second part of the problem consists then of developing the equations which relate these parameters to the rigid-body angular rates. After this is done, the transformation to the quaternion parameters will be performed.

To derive the equations from which θ and ℓ, m, n can be determined for a given p, q, r history, observe that the body attains its present attitude by precessing x, y z from their reference attitude—coincident with X, Y, Z—around $\mathbf{e}_\theta$ through θ. Each of the x, y, z axes thus generates portions of a cone as it precesses around $\mathbf{e}_\theta$. It follows therefore, that whatever direction cosines $\mathbf{e}_\theta$ makes with the fixed axes at any time t, it makes the same ones with the body axes at the same time. Thus, since l, m, n are the direction cosines of $\mathbf{e}_\theta$ with respect to X, Y, Z, they are also the direction cosines with respect to x, y, z. This implies the identity

$$\mathbf{e}_\theta = \mathbf{I}\ell + \mathbf{J}m + \mathbf{K}n \equiv \mathbf{i}\ell + \mathbf{j}m + \mathbf{k}n \tag{8}$$

Thus $\mathbf{e}_\theta$ is an interesting vector; at any time its components along the body and the reference axes are the same. Since Eq. (8) is an identity for all t, it may be differentiated with respect to t. Since $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ are fixed, while $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are in motion, this yields

$$\mathbf{I}\dot{\ell} + \mathbf{J}\dot{m} + \mathbf{K}\dot{n} = \dot{\mathbf{i}}\ell + \dot{\mathbf{j}}m + \dot{\mathbf{k}}n + \mathbf{i}\dot{\ell} + \mathbf{j}\dot{m} + \mathbf{k}\dot{n} \tag{9}$$

From Eq. (1)

$$\dot{\mathbf{i}} = \mathbf{j}r - \mathbf{k}q \tag{10}$$

and similarly for $\dot{\mathbf{j}}$, $\dot{\mathbf{k}}$. Substituting Eq. (10) in Eq. (9)

$$\mathbf{I}\dot{\ell} + \mathbf{J}\dot{m} + \mathbf{K}\dot{n} = \mathbf{i}(\dot{\ell} + qn - rm) + \mathbf{j}(\dot{m} + r\ell - pn) + \mathbf{k}(\dot{n} + pm - q\ell) \tag{11}$$

Equation (11) is a vector equation from which three scalar equations involving l, m, n and their derivatives will be obtained. These will be three (of the four) desired final equations. To obtain these, form the dot product of Eq. (11) successively with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Using the direction cosine matrix (7) to evaluate terms such as $\mathbf{i}\cdot\mathbf{I}$, $\mathbf{i}\cdot\mathbf{J}$, $\mathbf{j}\cdot\mathbf{I}$, etc. and noting that

$$\ell^2 + m^2 + n^2 = 1, \ell\dot{\ell} + m\dot{m} + n\dot{n} = 0 \tag{12}$$

three equations are obtained, of which the first is

$$\dot{\ell} + qn - rm = \dot{\ell}\cos\theta + (\dot{m}n - \dot{n}m)\sin\theta \tag{13}$$

Two other equations follow by cyclical permutation.

The fourth equation is obtained by noting that since the rotation θ occurs around $\mathbf{e}_\theta$, the projection of $\boldsymbol{\omega}$ on $\mathbf{e}_\theta$ must, in fact, be $\dot{\theta}$. Thus

$$\boldsymbol{\omega}\cdot\mathbf{e}_\theta = \dot{\theta} = p\ell + qm + rn \tag{14}$$

upon using Eqs. (1) and (4).

Equations (13) and (14) can be arranged into a standard form by solving for $\dot{\ell}$, $\dot{m}$, $\dot{n}$. After some manipulation, the final results are

$$\begin{aligned} \dot{\theta} &= p\ell + qm + rn \\ 2\dot{\ell} &= (rm - qn) + (p - \ell\dot{\theta})\cot(\theta/2) \\ 2\dot{m} &= (pn - r\ell) + (q - m\dot{\theta})\cot(\theta/2) \\ 2\dot{n} &= (q\ell - pm) + (r - n\dot{\theta})\cot(\theta/2) \end{aligned} \tag{15}$$

For prescribed time functions p, q, r, Eqs. (15) are the final four equations for determining ℓ, m, n, and θ. After computation, they can be substituted into the direction cosine matrix (7) thereby determining the attitude of x, y, z with respect to X, Y, Z at any time t.

In principle then, the attitude determination problem is solved. The practical difficulty may be a computational one, for Eqs. (15) contain $\cot(\theta/2)$. This would be computed as

$$\cot(\theta/2) = \sin\theta/(1 - \cos\theta) \tag{16}$$

so that $\sin\theta$, $\cos\theta$ have to be computed which may be burdensome for an airborne computer (for a real-time application). The other problem is an apparent singularity when $\theta = 0$, since $\cot 0 \to \infty$. However, the singularity can be resolved and it can be shown that, in the neighborhood of $\theta = 0$, Eqs. (15) should be replaced by

$$\dot{\ell} = \dot{m} = \dot{n} = 0, \dot{\theta} = (p^2 + q^2 + r^2)^{\frac{1}{2}} \tag{17a}$$

Correspondingly, ℓ, m, n take on the values

$$\ell = p/\dot{\theta}, m = q/\dot{\theta}, n = r/\dot{\theta} \tag{17b}$$

Now the quaternion scheme is obtained by making a transformation of variables that eliminates the cot function thus circumventing the difficulty at $\theta = 0$. Using Whittaker's notation, the quaternion components ξ, η, ζ, χ are defined as

$$\begin{aligned} \xi &= \ell\sin(\theta/2), \eta = m\sin(\theta/2) \\ \zeta &= n\sin(\theta/2), \chi = \cos(\theta/2) \end{aligned} \tag{18}$$

In terms of the quaternion parameters the direction cosine matrix (7) becomes

$$\begin{bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} \xi^2 - \eta^2 - \zeta^2 + \chi^2 & 2(\xi\eta + \zeta\chi) & 2(\xi\zeta - \eta\chi) \\ 2(\xi\eta - \zeta\chi) & -\xi^2 + \eta^2 - \zeta^2 + \chi^2 & 2(\eta\zeta + \xi\chi) \\ 2(\xi\zeta + \eta\chi) & 2(\eta\zeta - \xi\chi) & -\xi^2 - \eta^2 + \zeta^2 + \chi^2 \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{bmatrix} \tag{19}$$

and Eqs. (15) are converted to, in matrix form

$$\begin{bmatrix} \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \\ \dot{\chi} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \chi & -\zeta & \eta \\ \zeta & \chi & -\xi \\ \eta & \xi & \chi \\ \xi & -\eta & -\zeta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{20}$$

Equations (20) are the quaternion-rate equations. Observe that there are no singularities in these equations and no trigonometric functions. In fact, they are linear differential equations with time-varying coefficients. For specified time histories of p, q, r, Eqs. (20) are solved for ξ, η, ζ, χ and the instantaneous direction cosine matrix relating the moving frame x, y, z to the stationary frame X, Y, Z is then determined by Eq. (19). The results can also be generalized to the case where X, Y, Z has some known motion.

Using the previous results, it is possible now to solve explicitly a problem recently posed by Bar-Itzhack.[3] His problem is to determine initial values of the quaternion components given initial values of the direction cosine matrix relating the moving and fixed frames. His technique involves an iterative procedure, but this is not necessary since a direct solution is possible in the following way.

The Euler parameters ℓ, m, n, θ can be immediately related to the direction cosine elements using Eq. (7). Designate the elements by a_{ij}, then, adding the main diagonal elements produces

$$\cos\theta = (a_{11} + a_{22} + a_{33} - 1)/2 \tag{21a}$$

whereas subtracting the off-diagonal elements in pairs gives

$$\ell = (a_{23} - a_{32})/2 \sin\theta$$
$$m = (a_{31} - a_{13})/2 \sin\theta,\ n = (a_{12} - a_{21})/2 \sin\theta \tag{21b}$$

Now

$$\chi = \cos(\theta/2) = [(1 + \cos\theta)/2]^{1/2}$$
$$= (1 + a_{11} + a_{22} + a_{33})^{1/2}/2 \tag{22a}$$

Again

$$\xi = \ell \sin(\theta/2) = (a_{23} - a_{32}) \sin(\theta/2)/2 \sin \theta = (a_{23} - a_{32})/4\chi \tag{22b}$$

and similarly

$$\eta = m \sin(\theta/2) = (a_{31} - a_{13})/4\chi \tag{22c}$$

$$\zeta = n \sin(\theta/2) = (a_{12} - a_{21})/4\chi \tag{22d}$$

Equations (22) determine the quaternion components explicitly in terms of the direction cosines. Thus when initial values for the latter are known, the initial quaternion is determined. Of course, it was not necessary to work through the Euler axis and angle to derive Eqs. (22). Clearly, they also follow directly from the direction cosine matrix (19).

Conclusions

The quaternion scheme for rigid-body attitude determination has been developed starting with the Euler axis and angle. The axis/angle parameters, by themselves, provide a basis for determining attitude [Eqs. (7) and (15)], but they are not as "clean" computationally as the quaternion parameters [Eqs. (19) and (20)]. An explicit solution for the initial quaternion in terms of the initial values of the direction cosines has also been presented.

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Scanning the Celestial Sphere via Open-Loop Magnetic Control

G. M. CONNELL* AND J. M. HEDGEPETH†
Astro Research Corp., Santa Barbara, Calif.

FOR some satellite missions the capability of scanning the celestial sphere is essential. A scanning attitude behavior can be obtained by active control, or possibly by passive control such as an orbital regression-gravitational torque combination.[1] This Note presents a method of scanning with a spinning symmetric satellite in an equatorial circular orbit equipped with a current coil whose axis is coincident with the satellite spin axis. The interaction of the magnetic moment with the Earth's magnetic field causes precession of the spin axis. In addition, if the scan is initiated at the proper phase angle between the spin axis and the magnetic north pole, and if the current to the coil is actively controlled in the proper open-loop fashion, the precession of the spin axis (with a period of one day) is accompanied by a constant rate increase of the polar angle of the spin axis. The resulting scan pattern on the surface of the unit sphere is that of a "spherical helix," i.e., a linear increase of the polar angle with the angle of precession.

Discussion

The geometry used in the development is shown in Fig. 1. The basis denoted $\{\tilde{X}_\alpha\}$ is an inertial frame of reference with $\tilde{X}_1$ directed toward the first point of Aries and $\tilde{X}_3$ in the direction of the geographic North Pole. The angles λ and σ measure the polar location (elevation) and precession angle of the angular momentum vector $\tilde{H}$, respectively. The angles ψ, θ, and ϕ parametrize the motion of the satellite relative to the angular momentum vector in classical 3,1,3 Euler rotations. The state equations for the rotational motion of the satellite in a convenient form are[2]

$$\dot{h} = M_2\, s\theta \tag{1}$$

$$\dot{\theta} = (1/h)M_2\, c\theta \tag{2}$$

$$\dot{\lambda} = (1/h)(M_1\, c\psi - M_2\, c\theta\, s\psi) \tag{3}$$

$$\dot{\sigma} = (1/h\, s\lambda)(M_1\, s\psi + M_2\, c\theta\, c\psi) \tag{4}$$

$$\dot{\psi} = (h/I) - (1/h)[(\mathrm{ctn}\theta + s\psi\, \mathrm{ctn}\lambda)M_1 + c\theta\, c\psi\, \mathrm{ctn}\lambda M_2] \tag{5}$$

where $s \equiv$ sine, $c \equiv$ cosine, h is the magnitude of the angular momentum, I is the transverse moment of inertia of the satellite, and M_1 and M_2 are external torque components on transverse nonspinning body axes. (The torque component about the spin axis M_3 is zero.)

The components of the Earth's magnetic field (dipole model) at the satellite expressed in the $\{\tilde{X}_\alpha\}$ basis for a circular, equatorial orbit are[3]

$$B_1 = -(B_o/2)\, s\delta[sE - 3\, s(2\omega_o t - E)] \tag{6}$$

$$B_2 = -(B_o/2)\, s\delta[-cE + 3\, c(2\omega_o t - E)] \tag{7}$$

$$B_3 = B_o\, c\delta \tag{8}$$

where B_o is the field intensity at the orbital altitude, $\delta = 11°$ is the dipole tilt angle, and ω_o is the orbital frequency. The angle $E = \omega_E t + 3\pi/2 - E_o$, where ω_E is the earth rotation rate and the angle E_o is the longitudinal angle at $t = 0$ between the $\tilde{X}_1$ axis and the magnetic North Pole (79°N, 69°W, near Thule, Greenland[4]). The torque produced on the satellite is

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* Consultant; also Assistant Professor of Mechanical Engineering, University of California, Santa Barbara. Member AIAA.

† Vice-President. Associate Fellow AIAA.